Dedicated to Professor Hisaharu Umegaki on his seventieth birthday

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Abstract. The capacity region of \( m \)-user non-white Gaussian multiple access channel without feedback was exactly obtained. But it is not yet obtained in the feedback case without the upper bound of total capacity. We give the outer bound of capacity region in \( m \) user non-white Gaussian multiple access channel with noiseless and nodelayed feedback in general. Finally we remark that there are three types of outer bound of capacity region in the case of 2-user non-white Gaussian multiple access channel with feedback.

1. Introduction

The \( m \)-user Gaussian multiple access channel with feedback is defined by

\[
Y_i = \sum_{j=1}^{m} S_{ij} + Z_i, \quad i = 1, 2, \ldots
\]

For \( 1 \leq j \leq m \), \( S_j = \{S_{ij}; i = 1, 2, \ldots\} \) is a stochastic process representing input signal transmitted by the sender \( j \), where \( S_{ij} \) is input signal at the time \( i \) sent by the sender \( j \), \( Y = \{Y_i; i = 1, 2, \ldots\} \) is a stochastic process representing output signal, and \( Z = \{Z_i; i = 1, 2, \ldots\} \) is a non-degenerate, zero-mean Gaussian process representing noise, respectively. Though \( S_1, S_2, \ldots, S_m, Z \) are mutually independent in the non-feedback case, we have the different aspect in the feedback case. Let \( X_1, X_2, \ldots, X_m \) denote the input messages, where each \( X_j \) is uniformly distributed in \((1, 2, \ldots, 2^{nR_j})\) and is independent of the other messages. Since we have feedback, the input signal \( S_{ij} \) of sender \( j \) at time \( i \) is a function of the message \( X_j \) and the past values of the output signal \( Y^{i-1} = (Y_1, Y_2, \ldots, Y_{i-1}) \). For block length \( n \) we must specify a

\[
((2^{nR_1}, 2^{nR_2}, \ldots, 2^{nR_m}), n)
\]

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code with codewords
\[ S_{ij}^n(X_j, Y^{n-1}) = (S_{1j}(X_j), S_{2j}(X_j, Y^1), \ldots, S_{nj}(X_j, Y^{n-1})) , \]
\[ X_j \in \{1, 2, \ldots, 2^{nR_j}\}, \ j = 1, 2, \ldots, m. \]

In addition we require that the codewords satisfy the expected power constraints
\[ E\left[ \frac{1}{n} \sum_{i=1}^{n} S_{ij}^2(X_j, Y^{i-1}) \right] \leq P_j, \ j = 1, 2, \ldots, m, \]
where the expectation is taken over all possible noise sequences. The capacity of the non-white Gaussian channel without feedback was first characterized by Keilers [2]. In the \( m \)-user case the capacity region for \( n \) uses of the channel is the set of all rate vectors \((R_1, R_2, \ldots, R_m)\) satisfying
\[ \sum_{j \in \Gamma} R_j \leq \frac{1}{2n} \log \frac{\left| \sum_{j \in \Gamma} K_{S_j}^{(n)} + K_Z^{(n)} \right|}{|K_Z^{(n)}|}, \]
for every subset \( \Gamma \) of the senders \( \{1, 2, \ldots, m\} \), for some \( n \times n \) covariance matrices \( K_{S_j}^{(n)} \) of the vectors \( S_j = (S_{1j}, S_{2j}, \ldots, S_{nj}) \), satisfying the power constraint
\[ \frac{1}{n} Tr[K_{S_j}^{(n)}] \leq P_j, \ j = 1, 2, \ldots, m. \]

Keilers [2] used a sequential water filling procedure to obtain the extreme points of the convex hull of the capacity region. On the other hand the capacity region of \( m \)-user non-white Gaussian multiple access channel with feedback was given by Pombra and Cover [3] recently. Let \( \overline{\Gamma} \) denote the complement of the set \( \Gamma \). The capacity region is the set of all rate vectors \((R_1, R_2, \ldots, R_m)\) satisfying
\[ \sum_{j \in \Gamma} R_j \leq \frac{1}{2n} \log \frac{\left| K_{\sum_{j \in \Gamma} S_j - \sum_{j \in \Gamma} S_j + z}^{(n)} \right|}{|K_Z^{(n)}|}, \tag{1} \]
for every subset \( \Gamma \) of the senders \( \{1, 2, \ldots, m\} \), for some \( n \times n \) covariance matrices \( K_{\sum_{j \in \Gamma} S_j - \sum_{j \in \Gamma} S_j + z}^{(n)} \) of the vectors \( \sum_{j \in \Gamma} S_j - \sum_{j \in \Gamma} S_j + Z \), satisfying the power constraint
\[ \frac{1}{n} Tr[K_{S_j}^{(n)}] \leq P_j, \ j = 1, 2, \ldots, m. \tag{2} \]

The total capacity is defined as the maximum of \( \sum_{j=1}^{m} R_j \) under the power constraint (2). The total capacity \( C_n(\sum_{j=1}^{m} R_j) \) for \( m \)-user, non-white Gaussian multiple access channel without feedback is obtained by Pombra and Cover [3].
Theorem 1 (Cover and Pombra [3]).

\[
C_n(\sum_{j=1}^{m} R_j) = \frac{1}{2n} \sum_{j=1}^{n} \log (1 + \frac{(\lambda - \lambda_{i}^{(n)})^+}{\lambda_{i}^{(n)}}),
\]

where \((y)^+ = \max(y, 0)\), \(\{\lambda_{i}^{(n)}\}\) are eigenvalues of \(K_{Z}^{(n)}\), \(\lambda\) is chosen so that

\[
\sum_{i=1}^{n}(\lambda - \lambda_{i}^{(n)})^+ = n \sum_{j=1}^{m} P_j.
\]

It is difficult to obtain the total capacity \(C_{n,FB}(\sum_{j=1}^{m} R_j)\) for \(m\)-user non-white Gaussian multiple access channel with feedback exactly. But one of the upper bounds of total capacity is also obtained by Pombra and Cover [3].

Theorem 2 (Pombra and Cover [3]).

\[
C_n(\sum_{j=1}^{m} R_j) \leq C_{n,FB}(\sum_{j=1}^{m} R_j) \leq 2C_n(\sum_{j=1}^{m} R_j).
\]

In this paper we obtain the outer bound of capacity region in \(m\)-user non-white Gaussian multiple channel with feedback.

2. Outer Bound of Capacity Region

In order to obtain the upper bounds of the right hand side of (1), i.e.

\[
\frac{1}{2n} \log \frac{|K_{\sum_{j\in\Gamma} s_j - \sum_{j\in\bar{\Gamma}} s_j + Z}|}{|K_{Z}^{(n)}|}
\]

under the power constraint (2) for every subset \(\Gamma\) of the senders \(\{1, 2, \ldots, m\}\), we need the following lemmas. To avoid the complexity, we denote \(K_{S_j}^{(n)}, K_{Z}^{(n)}, \ldots\) by \(K_{S_j}, K_{Z}, \ldots\).

Lemma 1. The following assumptions (a) - (g) hold.

(a) \(Tr[K_{\sum_{j\in\Gamma} s_j}] \leq n(\sum_{j\in\Gamma} \sqrt{P_j})^2\)

(b) \(Tr[K_{\sum_{j\in\bar{\Gamma}} s_j}] \leq n(\sum_{j\in\bar{\Gamma}} \sqrt{P_j})^2\)

(c) \(Tr[K_{(\sum_{j\in\Gamma} s_j) (\sum_{j\in\bar{\Gamma}} s_j)}] \leq n(\sum_{j\in\Gamma} \sqrt{P_j})(\sum_{j\in\bar{\Gamma}} \sqrt{P_j})\)
(d) $\text{Tr}[K(\sum_{j\in\Gamma} s_j)z] \leq \sqrt{n}\left(\sum_{j\in\Gamma} \sqrt{P_j}\right)\sqrt{\text{Tr}[Kz]}$

(e) $\text{Tr}[K(\sum_{i\in\Gamma} s_i)z] \leq \sqrt{n}\left(\sum_{i\in\Gamma} \sqrt{P_i}\right)\sqrt{\text{Tr}[Kz]}$

(f) $\text{Tr}[K\sum_{j\in\Gamma} s_j + z] \leq (\sqrt{n}\sum_{j\in\Gamma} \sqrt{P_j} + \sqrt{\text{Tr}[Kz]})^2$

(g) $\text{Tr}[K\sum_{j\in\Gamma} s_j + z] \leq (\sqrt{n}\sum_{j\in\Gamma} \sqrt{P_j} + \sqrt{\text{Tr}[Kz]})^2$

**Proof.** Let $K_{S_iS_j}$ denote the cross-covariance matrix of $S_i$ and $S_j$.

$$\text{Tr}[K\sum_{j\in\Gamma} s_j] = \text{Tr}[\sum_{j\in\Gamma} Ks_j + \sum_{i\neq j} Ks_is_j]$$

$$= \sum_{j\in\Gamma} \text{Tr}[Ks_j] + 2\sum_{i<j} \text{Tr}[Ks_is_j]$$

$$\leq \sum_{j\in\Gamma} \text{Tr}[Ks_j] + 2\sum_{i<j} \sqrt{\text{Tr}[Ks_i]}\sqrt{\text{Tr}[Ks_j]}$$

$$\leq \sum_{j\in\Gamma} nP_j + 2\sum_{i<j} \sqrt{nP_i}\sqrt{nP_j}$$

$$= n\left(\sum_{j\in\Gamma} \sqrt{P_j}\right)^2.$$ 

Then we have (a). (b) is also given by the same reason as (a).

$$\text{Tr}[K(\sum_{j\in\Gamma} s_j)(\sum_{i\in\Gamma} s_i)]$$

$$\leq \sqrt{\text{Tr}[K\sum_{j\in\Gamma} s_j]}\sqrt{\text{Tr}[K\sum_{i\in\Gamma} s_i]}$$

$$\leq \sqrt{n\left(\sum_{j\in\Gamma} \sqrt{P_j}\right)^2}\sqrt{n\left(\sum_{i\in\Gamma} \sqrt{P_i}\right)^2}$$

$$= n\left(\sum_{j\in\Gamma} \sqrt{P_j}\right)(\sum_{i\in\Gamma} \sqrt{P_i}).$$

Then we have (c).

$$\text{Tr}[K(\sum_{j\in\Gamma} s_j)z]$$

$$\leq \sqrt{\text{Tr}[K\sum_{j\in\Gamma} s_j]}\sqrt{\text{Tr}[Kz]}$$

$$\leq \sqrt{n\left(\sum_{j\in\Gamma} \sqrt{P_j}\right)^2}\sqrt{\text{Tr}[Kz]}$$

$$= \sqrt{n\left(\sum_{j\in\Gamma} \sqrt{P_j}\right)}\sqrt{\text{Tr}[Kz]}.$$
Then we have (d). (e) is also given by the same reason as (d).

\[
\begin{align*}
    Tr[K \sum_{j \in \Gamma} s_j + z] & = Tr[K \sum_{j \in \Gamma} s_j + Kz + K(\sum_{j \in \Gamma} s_j)z + Kz(\sum_{j \in \Gamma} s_j)] \\
    & = Tr[K \sum_{j \in \Gamma} s_j] + Tr[Kz] + 2Tr[K(\sum_{j \in \Gamma} s_j)z] \\
    & \leq n(\sum_{j \in \Gamma} \sqrt{P_j})^2 + Tr[Kz] + 2\sqrt{n}(\sum_{j \in \Gamma} \sqrt{P_j})\sqrt{Tr[Kz]} \\
    & = (\sqrt{n} \sum_{j \in \Gamma} \sqrt{P_j} + \sqrt{Tr[Kz]})^2.
\end{align*}
\]

Then we have (f). (g) is also given by the same reason as (f). \(\square\)

We use the following notations.

\[
\begin{align*}
    A &= \frac{1}{2n} \log \frac{|K \sum_{j \in \Gamma} s_j + \sum_{j \in \Gamma} s_j + z|}{|Kz|} \\
    A_1 &= \frac{1}{2n} \log \frac{|K \sum_{j \in \Gamma} s_j - \sum_{j \in \Gamma} s_j + z|}{|Kz|} \\
    A_2 &= \frac{1}{2n} \log \frac{|K \sum_{j \in \Gamma} s_j - \sum_{j \in \Gamma} s_j + z|}{|Kz|}
\end{align*}
\]

\[
\begin{align*}
    a_1 &= \log\{(\sum_{j \in \Gamma} \sqrt{P_j} + \sqrt{1/n \cdot Tr[Kz]})^2 + (\sum_{j \in \Gamma} \sqrt{P_j})^2\} - \frac{1}{n} \log |Kz| \\
    a_2 &= \log\{(\sum_{j \in \Gamma} \sqrt{P_j} + \sqrt{1/n \cdot Tr[Kz]})^2 + (\sum_{j \in \Gamma} \sqrt{P_j})^2\} - \frac{1}{n} \log |Kz| \\
    \frac{b}{2} &= \log\{\sum_{j=1}^m \sqrt{P_j} + \sqrt{1/n \cdot Tr[Kz]}\} - \frac{1}{2n} \log |Kz| \\
    a &= \log\{(\sum_{j \in \Gamma} \sqrt{P_j} + \sum_{j \in \Gamma} \sqrt{P_j})^2 + \frac{1}{n} \cdot Tr[Kz]\} - \frac{1}{n} \log |Kz|
\end{align*}
\]

Lemma 2. \(A + A_1 \leq a_1.\)

Proof. By (b), (f) in Lemma 1,

\[
\begin{align*}
    A + A_1 & \leq \frac{1}{n} \log |K \sum_{j \in \Gamma} s_j + z + K \sum_{j \in \Gamma} s_j| - \frac{1}{n} \log |Kz| \\
    & \leq \log\{\frac{1}{n} \cdot Tr[K \sum_{j \in \Gamma} s_j + z] + \frac{1}{n} \cdot Tr[K \sum_{j \in \Gamma} s_j]\} - \frac{1}{n} \log |Kz| \\
    & \leq \log\{(\sum_{j \in \Gamma} \sqrt{P_j} + \sqrt{1/n \cdot Tr[Kz]})^2 + (\sum_{j \in \Gamma} \sqrt{P_j})^2\} - \frac{1}{n} \log |Kz| = a_1.
\end{align*}
\]
We have the result. \( \Box \)

**Lemma 3.** \( A + A_2 \leq a_2 \).

**Proof.** It is proved by the same method as Lemma 2. \( \Box \)

**Lemma 4.** \( A \leq \frac{b}{2} \).

**Proof.** By (a), (b), (c), (d), (e) in Lemma 1,

\[
A = \frac{1}{2n} \log |K| \sum_{i \in \Gamma} s_i + \sum_{j \in \Gamma} s_j + z - \frac{1}{2n} \log |K| \\
\leq \frac{1}{2n} \log \left\{ \frac{1}{n} \text{Tr}[K] \sum_{i \in \Gamma} s_i + \frac{1}{n} \text{Tr}[K] \sum_{j \in \Gamma} s_j + \frac{1}{n} \text{Tr}[K] \right\} \\
+ \frac{1}{n} \text{Tr}[K \sum_{i \in \Gamma} s_i \sum_{j \in \Gamma} s_j] + \frac{1}{n} \text{Tr}[K \sum_{i \in \Gamma} s_i z] \\
+ \frac{1}{n} \text{Tr}[K \sum_{j \in \Gamma} s_j \sum_{i \in \Gamma} s_i] + \frac{1}{n} \text{Tr}[K \sum_{j \in \Gamma} s_j z] + \frac{1}{n} \text{Tr}[K \sum_{i \in \Gamma} s_i] \\
+ \frac{1}{n} \text{Tr}[K \sum_{j \in \Gamma} s_j] \} - \frac{1}{2n} \log |K| \\
\leq \frac{1}{2} \log \left\{ \left( \sum_{i \in \Gamma} \sqrt{P_i} \right)^2 + \left( \sum_{j \in \Gamma} \sqrt{P_j} \right)^2 + \frac{1}{n} \text{Tr}[K] + 2 \left( \sum_{i \in \Gamma} \sqrt{P_i} \right) \left( \sum_{j \in \Gamma} \sqrt{P_j} \right) \\
+ 2 \left( \sum_{i \in \Gamma} \sqrt{P_i} \right) \left( \frac{1}{n} \text{Tr}[K] \right) + 2 \left( \sum_{j \in \Gamma} \sqrt{P_j} \right) \left( \frac{1}{n} \text{Tr}[K] \right) \} - \frac{1}{2n} \log |K| \\
= \frac{1}{2} \log \left\{ \left( \sum_{i \in \Gamma} \sqrt{P_i} + \sum_{j \in \Gamma} \sqrt{P_j} \right)^2 + 2 \left( \sum_{i \in \Gamma} \sqrt{P_i} \right) + \sum_{j \in \Gamma} \sqrt{P_j} \right) \left( \frac{1}{n} \text{Tr}[K] \right) \\
+ \frac{1}{n} \text{Tr}[K] \} - \frac{1}{2n} \log |K| \\
= \frac{1}{2} \log \left\{ \sum_{i \in \Gamma} \sqrt{P_i} + \sum_{j \in \Gamma} \sqrt{P_j} + \frac{1}{n} \text{Tr}[K] \right\}^2 - \frac{1}{2n} \log |K| = \frac{b}{2}.
\]

We have the result. \( \Box \)

**Lemma 5.** \( A_1 + A_2 \leq a \).

**Proof.** Since

\[
K \sum_{i \in \Gamma} s_i - \sum_{j \in \Gamma} s_j + z + K \sum_{j \in \Gamma} s_j - \sum_{i \in \Gamma} s_i + z = 2K \sum_{i \in \Gamma} s_i - \sum_{j \in \Gamma} s_j + 2Kz,
\]

it follows from (a), (b), (c) in Lemma 1 that

\[
A_1 + A_2 \\
\leq \frac{1}{n} \log |K \sum_{i \in \Gamma} s_i - \sum_{j \in \Gamma} s_j + Kz| - \frac{1}{n} \log |Kz| \\
\leq \log \left\{ \frac{1}{n} \text{Tr}[K \sum_{i \in \Gamma} s_i - \sum_{j \in \Gamma} s_j] + \frac{1}{n} \text{Tr}[Kz] \right\} - \frac{1}{n} \log |Kz|
\]
\begin{align*}
&= \log\left\{ \frac{1}{n} Tr[K \sum_{i \in \Gamma} S_i] \right\} + \frac{1}{n} Tr[K \sum_{j \in \bar{\Gamma}} S_j] - \frac{2}{n} Tr[K (\sum_{i \in \Gamma} S_i)(\sum_{j \in \bar{\Gamma}} S_j)] \\
&\quad + \frac{1}{n} Tr[K Z] - \frac{1}{n} \log |K Z| \\
&\leq \log\left\{ (\sum_{i \in \Gamma} \sqrt{P_i})^2 + (\sum_{j \in \bar{\Gamma}} \sqrt{P_j})^2 + 2(\sum_{i \in \Gamma} \sqrt{P_i})(\sum_{j \in \bar{\Gamma}} \sqrt{P_j}) + \frac{1}{n} Tr[K Z] \right\} \\
&\quad - \frac{1}{n} \log |K Z| \\
&= \log\left\{ (\sum_{i \in \Gamma} \sqrt{P_i} + \sum_{j \in \bar{\Gamma}} \sqrt{P_j})^2 + \frac{1}{n} Tr[K Z] \right\} - \frac{1}{n} \log |K Z| = a.
\end{align*}

We have the result. \(\square\)

Now we obtain the theorem.

**Theorem 3.** The outer bound of the capacity region is the set of all rate vectors \((R_1, R_2, \ldots, R_m)\) satisfying the following:

\[
\sum_{j=1}^{m} R_j \leq \min\left\{ \frac{b}{2}, a \right\},
\]

\[
2 \sum_{i \in \Gamma} R_i + \sum_{j \in \bar{\Gamma}} R_j \leq a_1,
\]

\[
\sum_{i \in \Gamma} R_i + 2 \sum_{j \in \bar{\Gamma}} R_j \leq a_2
\]

for every subset \(\Gamma\) of \(\{1, 2, \ldots, m\}\).

**Proof.** By definition

\[
\sum_{i \in \Gamma} R_i \leq A_1,
\]

\[
\sum_{j \in \Gamma} R_j \leq A_2,
\]

\[
\sum_{i=1}^{m} R_i \leq A.
\]

On the other hand it follows from Lemma 2 that

\[
A + A_1 \leq a_1.
\]

By Lemma 3

\[
A + A_2 \leq a_2.
\]

By Lemma 4

\[
A \leq \frac{b}{2}.
\]
By Lemma 5
\[ A_1 + A_2 \leq a. \tag{9} \]

Then it follows from (3),(4),(9) and (5),(8) that
\[ \sum_{i=1}^{m} R_i \leq \min \{ \frac{b}{2}, a \}. \]

And by (3),(5),(6)
\[ 2 \sum_{i \in \Gamma} R_i + \sum_{j \in \bar{\Gamma}} R_j \leq a_1. \]

And also by (4),(5),(7)
\[ \sum_{i \in \Gamma} R_i + 2 \sum_{j \in \bar{\Gamma}} R_j \leq a_2. \]

We have the result. \(\square\)

Finally we remark that there are three types of outer bounds of capacity region in 2-user non-white Gaussian multiple access channel with feedback. For simplicity we let \( m = 2, P_1 = P_2 = P \). Then we have
\[
\begin{align*}
a &= \log(4P + \frac{1}{n}Tr[R_Z]) - \frac{1}{n} \log |R_Z|, \\
\frac{b}{2} &= \log(2\sqrt{P} + \sqrt{\frac{1}{n}Tr[R_Z]}) - \frac{1}{2n} \log |R_Z|, \\
c &= a_1 = a_2 = \log \{ P + (\sqrt{P} + \sqrt{\frac{1}{n}Tr[R_Z]^2})^2 \} - \frac{1}{n} \log |R_Z|.
\end{align*}
\]

Then \( a \leq b \) and \( c \leq b \) are clear. If \( \frac{b}{2} \geq a \), then we have the following three cases.

(a) If \( c \leq a \) or \( \frac{2}{3}c \leq a \leq c \), then the outer bound is a tetragon.

(b) If \( \frac{c}{2} \leq a \leq \frac{2}{3}c \), then the outer bound is a pentagon.

(c) If \( \frac{c}{2} \geq a \), then the outer bound is a triangle.

If \( \frac{b}{2} \leq a \), then we have the following two cases.

(a) If \( c \leq \frac{b}{2} \) or \( \frac{2}{3}c \leq \frac{b}{2} \leq c \), then the outer bound is a tetragon.

(b) If \( \frac{b}{2} \leq \frac{2}{3}c \), then the outer bound is a pentagon.
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