Metric adjusted skew information, Metric adjusted correlation measure and Uncertainty relations

Kenjiro Yanagi

1 Metric adjusted skew information and Metric adjusted correlation measure

Inspired by the recent results in [4] and the concept of metric adjusted skew information introduced by Hansen in [6], we here give a further generalization for Schrödinger-type uncertainty relation applying metric adjusted correlation measure introduced in [6]. We firstly give some notations according to those in [4]. Let $M_n(\mathbb{C})$ and $M_{n,sa}(\mathbb{C})$ be the set of all $n \times n$ complex matrices and all $n \times n$ self-adjoint matrices, equipped with the Hilbert-Schmidt scalar product $\langle A, B \rangle = \text{Tr}[A^*B]$, respectively. Let $M_{n,+}(\mathbb{C})$ be the set of all positive definite matrices of $M_{n,sa}(\mathbb{C})$ and $M_{n,+;1}(\mathbb{C})$ be the set of all density matrices, that is $M_{n,+;1}(\mathbb{C}) \equiv \{ \rho \in M_{n,sa}(\mathbb{C}) | \text{Tr}\rho = 1, \rho > 0 \} \subset M_{n,+}(\mathbb{C})$.

Here $X \in M_{n,+}(\mathbb{C})$ means we have $\langle \phi | X | \phi \rangle \geq 0$ for any vector $|\phi\rangle \in \mathbb{C}^n$. In the study of quantum physics, we usually use a positive semidefinite matrix with a unit trace as a density operator $\rho$. In this section, we assume the invertibility of $\rho$.

A function $f : (0, +\infty) \rightarrow \mathbb{R}$ is said operator monotone if the inequalities $0 \leq f(A) \leq f(B)$ hold for any $A, B \in M_{n,sa}(\mathbb{C})$ such that $0 \leq A \leq B$. An operator monotone function $f : (0, +\infty) \rightarrow (0, +\infty)$ is said symmetric if $f(x) = xf(x^{-1})$ and normalized if $f(1) = 1$.

We represents the set of all symmetric normalized operator monotone functions by $\mathcal{F}_{\text{op}}$.

We have the following examples as elements of $\mathcal{F}_{\text{op}}$:

\begin{itemize}
  \item Example 1.1 ([6, 4, 2, 11]) \[ f_{RLD}(x) = \frac{2x}{x+1}, \quad f_{SLD}(x) = \frac{x+1}{2}, \quad f_{BKM}(x) = \frac{x-1}{\log x}, \]
  \item $f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2$, \quad $f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha-1)(x^{1-\alpha}-1)}$, $\alpha \in (0, 1)$.
\end{itemize}

*Division of Applied Mathematical Science, Graduate School of Science and Engineering, Yamaguchi University, 2-16-1, Tokiwadai, Ube 755-8611, Japan. yanagi@yamaguchi-u.ac.jp. The author was partially supported by the Japanese Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (C), 23540208
The functions $f_{BK}(x)$ and $f_{WD}(x)$ are normalized in the sense that $\lim_{x \to 1} f_{BK}(x) = 1$ and $\lim_{x \to 1} f_{WD}(x) = 1$. Note that a simple proof of the operator monotonicity of $f_{WD}(x)$ was given in [2]. See also [14] for the proof of the operator monotonicity of $f_{WD}(x)$ by use of majorization.

**Remark 1.2 ([4, 8, 10, 11])** For any $f \in \mathcal{F}_\text{op}$, we have the following inequalities:

\[
\frac{2x}{x + 1} \leq f(x) \leq \frac{x + 1}{2}, \quad x > 0.
\]

That is, all $f \in \mathcal{F}_\text{op}$ lies between the harmonic mean and the arithmetic mean.

For $f \in \mathcal{F}_\text{op}$ we define $f(0) = \lim_{x \to 0} f(x)$. We also denote the sets of regular and non-regular functions by

\[
\mathcal{F}_\text{op}^r = \{ f \in \mathcal{F}_\text{op} | f(0) \neq 0 \} \quad \text{and} \quad \mathcal{F}_\text{op}^n = \{ f \in \mathcal{F}_\text{op} | f(0) = 0 \}.
\]

**Definition 1.3 ([3, 4])** For $f \in \mathcal{F}_\text{op}^r$, we define the function $\tilde{f}$ by

\[
\tilde{f}(x) = \frac{1}{2} \left\{ (x + 1) - (x - 1) \frac{2f(0)}{f(x)} \right\}, \quad (x > 0).
\]

**Example 1.4**

\[
\tilde{f}_{WY}(x) = \sqrt{x}, \quad \tilde{f}_{WD}(x) = \frac{1}{2} \left\{ x + 1 - (x^a - 1)(x^{1-a} - 1) \right\} = \frac{x^a + x^{1-a}}{2}, \quad \tilde{f}_{SLD}(x) = \frac{2x}{x + 1},
\]

Then we have the following theorem.

**Theorem 1.5 ([3, 2, 12])** The correspondence $f \to \tilde{f}$ is a bijection between $\mathcal{F}_\text{op}^r$ and $\mathcal{F}_\text{op}^n$.

We can use matrix mean theory introduced by Kubo-Ando in [8]. Then a mean $m_f$ corresponds to each operator monotone function $f \in \mathcal{F}_\text{op}$ by the following formula

\[
m_f(A, B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},
\]

for $A, B \in M_{n,+}$. By the notion of matrix mean, we may define the set of the monotone metrics [9] by the following formula

\[
\langle A, B \rangle_{\rho, f} = Tr [ A m_f(L_{\rho}, R_{\rho})^{-1}(B) ],
\]

where $L_{\rho}(A) = \rho A$ and $R_{\rho}(A) = A \rho$.

**Definition 1.6 ([6, 3])** For $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+1}(\mathbb{C})$ and $f \in \mathcal{F}_\text{op}^r$, we define the following quantities:

\[
\text{Corr}_{\rho}^f(A, B) \equiv \frac{f(0)}{2} \langle i[\rho, A], i[\rho, B] \rangle_{\rho, f}, \quad I_{\rho}^f(A) \equiv \text{Corr}_{\rho}^f(A, A),
\]

\[
C_{\rho}^f(A, B) \equiv Tr [ m_f(L_{\rho}, R_{\rho})(A) B ], \quad C_{\rho}^f(A) \equiv C_{\rho}^f(A, A),
\]

\[
U_{\rho}^f(A) \equiv \sqrt{V_{\rho}(A)^2 - (V_{\rho}(A) - I_{\rho}^f(A))^2}.
\]
The quantity \( I^c_\rho(A) \) is known as metric adjusted skew information [6]. It is notable that the metric adjusted correlation measure \( Corr^c_\rho(A, B) \) was firstly introduced in [6] for a regular Morozova-Chentsov function \( c \). Recently the notation \( I^c_\rho(A, B) \) in [1] and the notation \( I^c_\rho(A, B) \) in [5] were used. In addition, it is useful for the readers to be noted that the metric adjusted correlation measure \( Corr^c_\rho(A, B) \) can be expressed as a difference of covariances [5]. Throughout the present paper, we use the notation \( Corr^c_\rho(A, B) \) as the metric adjusted correlation measure, to avoid the confusion of the readers. Then we have the following proposition.

**Proposition 1.7 ([3, 4])** For \( A, B \in M_{n,sa}(\mathbb{C}) \), \( \rho \in M_{n,+1}(\mathbb{C}) \) and \( f \in F_{op}^r \), we have the following relations, where we put \( A_0 \equiv A - Tr[\rho A]I \) and \( B_0 \equiv B - Tr[\rho B]I \).

1. \( I^c_\rho(A) = Tr[\rho A_0^2] - Tr[m_f(L, R)(A_0)A_0] = V_\rho(A) - C_\rho(A_0). \)
2. \( J^c_\rho(A) = Tr[\rho A_0^2] + Tr[m_f(L, R)(A_0)A_0] = V_\rho(A) + C_\rho(A_0). \)
3. \( 0 \leq I^c_\rho(A) \leq U^c_\rho(A) \leq V_\rho(A). \)
4. \( U^c_\rho(A) = \sqrt{I^c_\rho(A) J^c_\rho(A)}. \)
5. \( Corr^c_\rho(A, B) = \frac{1}{2} Tr[\rho A_0 B_0] + \frac{1}{2} Tr[\rho B_0 A_0] - \frac{1}{2} Tr[m_f(L, R)(A_0)B_0] - \frac{1}{2} Tr[m_f(L, R)(B_0)A_0] - \frac{1}{2} Tr[\rho A_0 B_0] - \frac{1}{2} Tr[\rho B_0 A_0] - C_\rho(A_0, B_0). \)

## 2 Uncertainty relations

**Theorem 2.1** For \( f \in F_{op}^r \), if we have

\[
\frac{x+1}{2} + \hat{f}(x) \geq 2f(x),
\]

then we have

\[
U^c_\rho(A)U^c_\rho(B) \geq f(0)|Tr[\rho[A, B]]|^2,
\]

\[
U^c_\rho(A)U^c_\rho(B) \geq 4f(0)|Corr^c_\rho(A, B)|^2.
\]

for \( A, B \in M_{n,sa}(\mathbb{C}) \) and \( \rho \in M_{n,+1}(\mathbb{C}) \).

If we use the function

\[
f_{WYD}(x) = \alpha(1 - \alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \quad \alpha \in (0, 1),
\]

then we obtain the following uncertainty relation.

**Corollary 2.2** For \( A, B \in M_{n,sa}(\mathbb{C}) \) and \( \rho \in M_{n,+1}(\mathbb{C}) \), we have

\[
U^{f_{WYD}}_\rho(A)U^{f_{WYD}}_\rho(B) \geq \alpha(1 - \alpha)|Tr[\rho[A, B]]|^2,
\]

\[
U^{f_{WYD}}_\rho(A)U^{f_{WYD}}_\rho(B) \geq 4\alpha(1 - \alpha)|Corr_{\rho, \alpha, \frac{1}{2}}(A, B)|^2,
\]

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where
\[
\text{Corr}_{\rho, \alpha, \frac{1}{2}}(A, B) = \frac{1}{2} Tr[\rho AB] + \frac{1}{2} Tr[\rho BA] - \frac{1}{2} Tr[\rho^\alpha A \rho^{1-\alpha} B] - \frac{1}{2} Tr[\rho^{1-\alpha} A \rho^\alpha B].
\]

Now we show that \(f_{WYD}(x)\) satisfies (1). By Lemma 3.3 in [15] we have for \(0 \leq \alpha \leq 1\) and \(x > 0\),
\[
(1 - 2\alpha)^2(x - 1)^2 - (x^\alpha - x^{1-\alpha})^2 \geq 0.
\]
This inequality can be rewritten by
\[
(x^{2\alpha} - 1)(x^{2(1-\alpha)} - 1) \geq 4\alpha(1 - \alpha)(x - 1)^2.
\]
Thus we have
\[
\frac{x + 1}{2} + f_{WYD}(x) = x + 1 - \frac{1}{2}(x^\alpha - 1)(x^{1-\alpha} - 1)
\]
\[
= \frac{1}{2}(x^{\alpha} + 1)(x^{1-\alpha} + 1)
\]
\[
\geq 2\alpha(1 - \alpha)\frac{(x - 1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}
\]
\[
= 2f_{WYD}(x).
\]

When we don’t assume the condition (1) in Theorem 2.1, we obtain the following:

**Theorem 2.3** ([4]) For \(f \in F_{op}\), we have
\[
U^f_{\rho}(A)U^f_{\rho}(B) \geq f(0)^2||Tr[\rho [A, B]]||^2,
\]
\[
U^f_{\rho}(A)U^f_{\rho}(B) \geq 4f(0)^2||\text{Corr}^f_{\rho}(A, B)||^2
\]
for \(A, B \in M_{n,sa}(\mathbb{C})\) and \(\rho \in M_{n,+}(\mathbb{C})\).

**References**


